JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **163**, 113–135 (1992)

Inversion of Integral Transforms Associated with a Class of Perturbed Heat Equations

AHMED I. ZAYED*

California Polytechnic State University,
San Luis Obispo, California 93407

AND

DEBORAH TEPPER HAIMO

University of Missouri,
St. Louis, Missouri 63121-4499

Submitted by Bruce C. Berndt

Received July 25, 1989

Let L_x be the Sturm–Liouville differential operator $L_x = -d^2/dx^2 + q(x)$; $x \in (0, \infty)$. We assume that L_x has either a purely discrete spectrum that is bounded from below by zero or a continuous spectrum that fills up the interval $(0, \infty)$ with, possibly, a finite number of negative eigenvalues. The W-transform $\tilde{\psi}(x)$ of $\psi(y) \in L^2(0, \infty)$ is defined by

$$\tilde{\psi}(x) = \int_0^\infty \psi(y)\, g(x, y; 1)\, dy,$$

where $g(x, y; t)$ is a function associated with the fundamental solution of the perturbed equation

$$-L_x u(x, t) = \frac{\partial u(x, t)}{\partial t}.$$

The main purpose of this paper is to derive an inversion formula for the W-transform. This inversion formula generalizes the known inversion formulae for the Weierstrass, Weierstrass–Hankel convolution, and Weierstrass–Laguerre transforms. The results of this paper are easily extended to the case where L_x is considered over the entire line $(-\infty, \infty)$. © 1992 Academic Press, Inc.

1. INTRODUCTION

Solving certain boundary value problems involving heat equations of different types leads to integral transforms whose kernels are functions

* Current address: Department of Mathematics, University of Central Florida, Orlando, FL 32816.

0022-247X/92 $3.00

associated with the fundamental solutions of these heat equations. Various properties of these integral transforms, as well as their inversion formulae, have been studied by seveal people in recent years.

In [12], for example, I. I. Hirschman and D. V. Widder developed an inversion and representation theory for integral transforms whose kernels are associated with the fundamental solution of the classical heat equation

$$u_{xx}(x, t) = u_t(x, t). \tag{1.1}$$

F. M. Cholewinski and D. T. Haimo obtained similar results for integral transforms whose kernels are associated with the generalized heat equation

$$u_{xx} + \frac{2\nu}{x} u_x = u_t, \qquad \nu > 0, \tag{1.2}$$

[1, 5] and the Laguerre differential heat equation

$$x u_{xx} + (\alpha + 1 - x)\, u_x = u_t, \qquad \alpha > -1, \tag{1.3}$$

[2, 3]. Among other results, they derived inversion formulae for the Weierstrass–Hankel convolution transform and the dual Weierstrass–Laguerre transform. The former is related to Eq. (1.2) and the latter to Eq. (1.3). Other related transforms were also studied by D. T. Haimo in [4, 10, 11].

In a recent paper [20], we introduced an integral transform which we called the W-transform whose kernel is associated with the generalized heat equation

$$-L_x u(x, t) = u_t(x, t), \tag{1.4}$$

where L_x is a self-adjoint differential operator of order $2N$ which has, among other restrictions, a purely discrete spectrum. The kernel of the W-transform is related to Eq. (1.4) in the same way that the kernels of the Weierstrass–Hankel convolution transform and the dual Weierstrass–Laguerre transform are related to Eqs. (1.2) and (1.3), respectively. An inversion formula for the W-transform was also found in [20].

The main goal of this paper is to derive an inversion formula for the W-transform associated with the perturbed heat equation

$$-L_x u(x, t) = u_t(x, t),$$

where L_x is the Sturm–Liouville differential operator

$$L_x = -\frac{d^2}{dx^2} + q(x),$$

and $q(x)$ is continuous.

Unlike our investigation in [20], we not only impose fewer restrictions on the operator L_x, but also extend our results to the case where L_x has a continuous spectrum. The results of this paper generalize those previously obtained by Hirschman and Widder [12] and Cholewinski and Haimo [1, 11].

2. Preliminaries

Consider the second-order differential equation

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+l(x)\,y=\lambda r(x)\,y, \tag{2.1}$$

where $p(x)$, $r(x)$ are positive, $p(x)$ has a continuous first derivative, $p(x)\,r(x)$ has a continuous second derivative, and λ is a complex number. By means of the substitutions

$$z=\int\left(\frac{r(x)}{p(x)}\right)^{1/2}dx, \qquad u=(r(x)\,p(x))^{1/4}\,y,$$

Eq. (2.1) takes the form

$$-\frac{d^2u}{dz^2}+q(z)\,u=\lambda u, \tag{2.2}$$

where $q(z)=\theta''(z)/\theta(z)+l(x)/r(x)$ and $\theta(z)=(r(x)\,p(x))^{1/4}$.

Let L denote the Sturm–Liouville differential operator

$$L=-\frac{d^2}{dz^2}+q(z). \tag{2.3}$$

Then, Eq. (2.2) takes the form

$$Lu=\lambda u.$$

We consider the singular Sturm–Liouville problem

$$Lu(x,\lambda)=\lambda u(x,\lambda), \qquad 0<x<\infty, \tag{2.4}$$

with the boundary conditions

$$(1)\quad |u(x,\lambda)|<\infty \tag{2.5}$$

$$(2)\quad u(0,\lambda)\cos\alpha+u'(0,\lambda)\sin\alpha=0, \qquad \alpha\in[0,2\pi], \tag{2.6}$$

where $q(x)$ is assumed to be continuous and real-valued on $[0, \infty)$. From now on we denote the singular Sturm–Liouville problem on $[0, \infty)$ by SS − L.

Let $\phi(x, \lambda)$ be the particular solution of the SS − L problem satisfying

$$\begin{aligned} \phi(0, \lambda) &= \sin \alpha, \\ \phi'(0, \lambda) &= -\cos \alpha. \end{aligned} \tag{2.7}$$

It is well known, [14, 15], that there exists a nondecreasing function ρ such that, for any $f(x) \in L^2(0, \infty)$, the generalized Fourier transform $\hat{f}(\lambda)$ of $f(x)$ is defined by

$$\begin{aligned} \hat{f}(\lambda) &= \lim_{n \to \infty} \int_0^n f(x)\, \phi(x, \lambda)\, dx \\ &= \int_0^\infty f(x)\, \phi(x, \lambda)\, dx \end{aligned} \tag{2.8}$$

in $L^2\{(-\infty, \infty), d\rho(\lambda)\}$. Moreover, the Sturm–Liouville eigenfunction expansion of f given by

$$f(x) \sim \int_{-\infty}^{\infty} \hat{f}(\lambda)\, \phi(x, \lambda)\, d\rho(\lambda) \tag{2.9}$$

converges to $f(x)$ in $L^2(0, \infty)$.

If $f(x)$ and $\hat{f}(\lambda)$ are related as above, then the correspondence

$$f \leftrightarrow \hat{f}$$

is a one–one, norm-preserving mapping of the space $L^2(0, \infty)$ into the space $L^2\{(-\infty, \infty), d\rho(\lambda)\}$; in fact if

$$f \leftrightarrow \hat{f}, \qquad g \leftrightarrow \hat{g},$$

then we have Parseval's equality

$$\int_0^\infty f(x)\, g(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(\lambda)\, \hat{g}(\lambda)\, d\rho(\lambda), \tag{2.10}$$

and, in particular,

$$\int_0^\infty |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2\, d\rho(\lambda). \tag{2.11}$$

If $f(x)$ is continuous on $(0, \infty)$, and the integral

$$\int_{-\infty}^{\infty} \hat{f}(\lambda)\, \phi(x, \lambda)\, d\rho(\lambda)$$

converges absolutely and uniformly with respect to x in every finite interval, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda)\, \phi(x, \lambda)\, d\rho(\lambda), \qquad x \in (0, \infty). \tag{2.12}$$

Let $\mathcal{M}$ be the class of all functions $f(x) \in L^2(0, \infty)$ such that

(i) $Lf(x) \in L^2(0, \infty)$,

(ii) $f(0) \cos \alpha + f'(0) \sin \alpha = 0$,

(iii) $\lim_{x \to \infty} W\{\phi(x, \lambda), f(x)\} = 0$ for all λ in the spectrum, where W is the Wronskian.

If $q(x)$ is summable over every finite interval and $f(x) \in \mathcal{M}$, then the integral in (2.12) converges to $f(x)$ absolutely for any x, $0 \leqslant x < \infty$, provided that $f(x)$ is of bounded variation [14, pp. 319, 347].

Throughout the paper, we assume that the support of $d\rho(\lambda)$ is bounded from below. More precisely, we assume that $q(x)$ is either

(A) non-decreasing, convex, and $q(x) \to \infty$ as $x \to \infty$,

or

(B) $(1 + x^2)\, q(x) \in L^1(0, \infty)$.

Under these assumptions, the SS $-$ L problem is known to have in case (A), [15, Sect. 7.1],

(A*) a purely discrete spectrum; its eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ have infinity as their sole limit point, and a lower bound which, without loss of generality, may be taken as zero, and in case (B) [14, pp. 198, 209],

(B*) a continuous spectrum that fills up the half-line $(0, \infty)$ and has, possibly, a finite number of negative eigenvalues.

Similar assumptions hold for the whole line. In the analogue of condition (A), for example, $q(x)$ must be non-increasing in $(-\infty, 0)$, and $q(x) \to \infty$ as $x \to -\infty$ [15, Sect. 5.14].

In case (A), if we denote the orthonormalized eigenfunctions by $\{\phi_n(x)\}_{n=0}^{\infty}$, then formulae (2.8), (2.9), and (2.11) take the form

$$\hat{f}(n) \sim \int_0^{\infty} f(x)\, \phi_n(x)\, dx, \tag{2.13}$$

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)\, \phi_n(x), \tag{2.14}$$

$$\int_0^{\infty} |f(x)|^2\, dx = \sum_{n=0}^{\infty} |\hat{f}(n)|^2. \tag{2.15}$$

In case (B), formulae (2.8) and (2.11) are unchanged, but (2.9) may take the form

$$\begin{aligned} f(x) &\sim \sum_{\lambda_n \in \wedge} \hat{f}(\lambda_n)\, \phi(x, \lambda_n)\, d_n + \int_0^{\infty} \hat{f}(\lambda)\, \phi(x, \lambda)\, d\rho(\lambda) \\ &= \int_{-b}^{\infty} \hat{f}(\lambda)\, \phi(x, \lambda)\, d\rho(\lambda), \end{aligned} \tag{2.16}$$

where $\wedge$ is the set of negative eigenvalues,

$$\begin{aligned} d_n &= \rho(\lambda_n^+) - \rho(\lambda_n^-), \\ -b &= \min\{\lambda_n \in \wedge\}. \end{aligned} \tag{2.17}$$

We note further that, in case (A), $\phi_n(x) = \phi(x, \lambda_n)$ is uniformly bounded in x as $n \to \infty$ [15, Sect. 8.4], whereas in case (B), as $\lambda \to \infty$, if $\sin \alpha = 0$,

$$\phi(x, \lambda) = -\frac{\sin \sqrt{\lambda}\, x}{\sqrt{\lambda}} \cos \alpha + 0\left(\frac{1}{\lambda}\right), \tag{2.18}$$

$$\rho'(\lambda) = 0(\sqrt{\lambda}), \tag{2.19}$$

and, if $\sin \alpha \neq 0$,

$$\phi(x, \lambda) = \cos(\sqrt{\lambda}\, x) \sin \alpha + 0\left(\frac{1}{\sqrt{\lambda}}\right), \tag{2.20}$$

$$\rho'(\lambda) = 0\left(\frac{1}{\sqrt{\lambda}}\right). \tag{2.21}$$

These estimates hold uniformly in $x \in [0, \infty)$ [6, Lemmas 1, 2].

Further, if $q(x)$ belongs to $C^k(0, \infty)$, then so does $\phi(x, \lambda)$ and it can be shown that

$$\phi^{(j)}(x, \lambda) = O(\lambda^p) \qquad \text{as} \quad \lambda \to \infty, \tag{2.22}$$

for $j = 0, 1, ..., k$ and some $p > 0$, uniformly for $x \in [0, \infty)$. In particular,

$$|\phi(x, \lambda)| \leqslant c, \qquad \text{for all } x, \lambda, \tag{2.23}$$

for some $c > 0$, whereas, for any fixed complex number z,

$$\phi(z, \lambda) = O(e^{\sqrt{\lambda}\,|z|}), \tag{2.24}$$

as $\lambda \to \infty$ [17, Lemma 2.1, 19].

3. The Kernel $g(x, y; t)$

Consider $g(x, y; t)$ defined by

$$g(x, y; t) = \int_{-\infty}^{\infty} e^{-\lambda t} \phi(x, \lambda)\, \phi(y, \lambda)\, d\rho(\lambda), \qquad \operatorname{Re} t > 0. \tag{3.1}$$

Since the support of $d\rho(\lambda)$ is bounded from below, the lower limit of the integral in (3.1) is a finite number $-b = \inf\{\operatorname{supp}(d\rho)\}$. We note that $g(x, y; t)$ is well defined and $g(x, y; t) = g(y, x; t)$. In fact, as a consequence of the estimates of the previous section, it follows that $g(x, y; t)$ is analytic in t for $\operatorname{Re} t \geqslant \delta > 0$ since its defining integral is absolutely and uniformly convergent in that region. By using the same argument as above and appealing to (2.22), we can show that if $q(x)$ is analytic in some complex domain D, then so is $g(x, y; t)$ as a function of x and y. Thus we henceforth assume that $q(x)$ is analytic except possibly at the origin.

We introduce next some additional properties of $g(x, y; t)$.

Lemma 3.1.

(i) *For fixed x and t, $g(x, y; t)$ is in $L^2(0, \infty)$* (3.2)

(ii) $$\int_0^{\infty} g(x, y; t)\, \phi(y, \lambda)\, dy = e^{-\lambda t} \phi(x, \lambda). \tag{3.3}$$

(iii) $$\int_0^{\infty} g(x, z; t_1)\, g(z, y; t_2)\, dz = g(x, y; t_1 + t_2). \tag{3.4}$$

Proof. (i) (ii) For fixed x and t, let

$$g_{x,t}(y) = g(x, y; t),$$
$$h_{x,t}(\lambda) = e^{-\lambda t} \phi(x, \lambda).$$

By (2.23) and the fact that $\rho'(\lambda)$ is at most $O(\sqrt{\lambda})$ as $\lambda \to \infty$, we note that $h_{x,t}(\lambda) \in L^2\{(-\infty, \infty), d\rho\}$. From (3.1) and (2.9) we conclude that $h_{x,t}(\lambda)$ is the generalized Fourier transform of $g_{x,t}(y)$, so that

$$h_{x,t}(\lambda) = \hat{g}_{x,t}(\lambda), \tag{3.5}$$

and hence (2.8) yields (3.3). Appealing to (2.11), we obtain

$$\begin{aligned}\int_0^\infty |g_{x,t}(y)|^2\,dy &= \int_0^\infty |g(x,y;t)|^2\,dy\\ &= \int_{-\infty}^\infty e^{-2\lambda t}\,|\phi(x,\lambda)|^2\,d\rho(\lambda)\\ &\leqslant c^2 \int_{-b}^\infty e^{-2\lambda t}\,d\rho(\lambda)\\ &< \infty,\end{aligned}$$

proving (3.2).

(iii) The integral in (3.4) exists since

$$\left|\int_0^\infty g(x,z;t_1)\,g(z,y;t_2)\,dz\right|^2 \leqslant \left(\int_0^\infty |g(x,z;t_1)|^2\,dz\right)\left(\int_0^\infty |g(z,y;t_2)|^2\,dz\right)$$

by (i). As in the proof of part (i), we can show that

$$\hat{g}_{x,t_1}(\lambda) = e^{-\lambda t_1}\phi(x,\lambda), \qquad \hat{g}_{y,t_2}(\lambda) = e^{-\lambda t_2}\phi(y,\lambda).$$

Now an application of Parseval's equality (2.10) yields

$$\begin{aligned}\int_0^\infty g(x,z;t_1)\,g(z,y;t_2)\,dz &= \int_{-\infty}^\infty e^{-\lambda t_1}\phi(x,\lambda)\,e^{-\lambda t_2}\phi(y,\lambda)\,d\rho(\lambda)\\ &= \int_{-\infty}^\infty e^{-\lambda(t_1+t_2)}\phi(x,\lambda)\,\phi(y,\lambda)\,d\rho(\lambda)\\ &= g(x,y;t_1+t_2). \qquad \text{Q.E.D.}\end{aligned}$$

For our principal result, we need a subtraction formula corresponding to the addition formula (3.4). To this end, since we cannot establish directly an analogue of (3.3), we introduce it as an assumption for the following lemma.

LEMMA 3.2. *Let there exist a bounded, continuous function $\eta(x, y, t)$ such that, for some complex number a,*

$$\int_0^\infty g(ax,y;t)\,\eta(x,y,t)\,\phi(ay,\lambda)\,dy = e^{\lambda t}\phi(x,\lambda), \qquad \operatorname{Re} t > 0, \tag{3.6}$$

where the integral converges absolutely. Then, for $0 < t_1 < t_2$,

$$\int_0^\infty g(ax,z;t_1)\,g(az,y;t_2)\,\eta(x,z,t_1)\,dz = g(x,y;t_2-t_1). \tag{3.7}$$

Proof. The integral in (3.7) exists since η is bounded and $g \in L^2(0, \infty)$. By combining (3.7), (3.1), and (3.6), we obtain

$$\begin{aligned}
&\int_0^\infty g(ax, z; t_1)\, g(az, y; t_2)\, \eta(x, z, t_1)\, dz \\
&= \int_0^\infty g(ax, z; t_1)\, \eta(x, z, t_1)\, dz \int_{-\infty}^\infty e^{-\lambda t_2} \phi(az, \lambda)\, \phi(y, \lambda)\, d\rho(\lambda) \\
&= \int_{-\infty}^\infty e^{-\lambda t_2} \phi(y, \lambda)\, d\rho(\lambda) \int_0^\infty g(ax, z; t_1)\, \eta(x, z, t_1)\, \phi(az, \lambda)\, dz \\
&= \int_{-\infty}^\infty e^{-\lambda(t_2 - t_1)} \phi(y, \lambda)\, \phi(x, \lambda)\, d\rho(\lambda) \\
&= g(x, y; t_2 - t_1).
\end{aligned}$$

The interchange of the order of the integrations is permissible by Fubini's theorem in view of the absolute convergence of the integrals involved.

Q.E.D.

Let us observe that $g(x, y; t)$, as a function of x or y, is in $L^2(0, \infty)$ and satisfies the differential equations

$$\frac{\partial g}{\partial t} = -L_x g, \qquad \frac{\partial g}{\partial t} = -L_y g, \tag{3.8}$$

with

$$\begin{aligned}
&g(0, y; t) \cos \alpha + g'(0, y; t) \sin \alpha = 0 \\
&g(x, 0; t) \cos \alpha + g'(x, 0; t) \sin \alpha = 0,
\end{aligned} \tag{3.9}$$

and

$$g(x, y; 0) = \delta(x - y).$$

Moreover, we know that

$$\int_0^\infty g(x, y; t)\, \phi(y, \lambda)\, dy = e^{-\lambda t} \phi(x, \lambda),$$

and let

$$e^{-\lambda t} \phi(x, \lambda) = F(x, \lambda, t).$$

In the special case where $a = i$ and $q(ix) = -q(x)$, e.g., $q(z) = \tilde{q}(z^2)$, where $\tilde{q}$ is odd, we show how $\eta(x, y, t)$ may be constructed. It is easy to see that the transformation $x \to ix$ takes the differential equation $L_x u = \lambda u$ into $L_x u = -\lambda u$. It follows that the transformation $t \to -t$, $x \to ix$, $y \to iy$ leaves the differential equations (3.8) invariant and hence $g(ix, iy; -t)$ is also a solution of (3.8) satisfying conditions similar to (3.9). Thus there exists a constant C such that $g(x, y; t) = Cg(ix, iy; -t)$. Similarly, $g(x, iy; -t)$ is a solution of the equations $\partial u/\partial t = -L_y u$ and $\partial u/\partial t = L_x u$. If $g(x, iy; -t)$ is in $L^2(0, \infty)$ as a function of y, then we take $\eta(x, y, t) = C(g(x, iy; -t)/g(ix, y; t))$, since, in this case,

$$\begin{aligned}\int_0^\infty g(ix, y; t)\, \eta(x, y, t)\, \phi(iy, \lambda)\, dy &= C \int_0^\infty g(x, iy; -t)\, \phi(iy, \lambda)\, dy \\ &= \tilde{F}(x, \lambda, t).\end{aligned}$$

But $\tilde{F}(x, \lambda, -t)$ is easily seen to satisfy the same differential equation and boundary conditions as $F(x, \lambda, t)$; namely,

$$\frac{\partial u}{\partial t} = -L_x u,$$

$$F(0, \lambda, t) \cos \alpha + F'(0, \lambda, t) \sin \alpha = 0$$

$$\lim_{t \to 0} \tilde{F}(x, \lambda, -t) = \lim_{t \to 0} F(x, \lambda, t).$$

Therefore,

$$\begin{aligned}\tilde{F}(x, \lambda, -t) &= F(x, \lambda, t), \\ &= e^{-\lambda t} \phi(x, \lambda).\end{aligned}$$

4. Inversion Theorem

In this section we introduce the P-transform. A special case, the W-transform, generalizes the Weierstrass transform, the Weierstrass–Hankel convolution transform, and the dual Weierstrass–Laguerre transform given in [16, 1, 11], respectively.

The primary goal of this section is to derive an inversion formula for the W-transform.

Consider the parabolic partial differential equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} - q(x)\, u(x, t) = -L_x u(x, t), \tag{4.1}$$

which reduces to the classical heat equation when $q(x) = 0$. The subscript x in (4.1) is used to indicate that the operator L is acting on $u(x, t)$ as a function of x. This notation is adopted throughout this article. We call (4.1) the perturbed heat equation.

In the previous section, we introduced $g(x, y; t)$ which is now used to solve the following problem:

If $f(x) \in L^2(0, \infty)$ and satisfies (2.5) and (2.6), find $f(x, t)$ such that

$$\frac{\partial f(x, t)}{\partial t} = -L_x f(x, t) \tag{4.2}$$

with

$$\begin{gathered} f(x, 0) = f(x), \\ f(0, t) \cos \alpha + f'(0, t) \sin \alpha = 0, \\ |f(x, t)| < \infty. \end{gathered} \tag{4.3}$$

To this end, set

$$f(x, t) = \int_0^\infty f(y)\, g(x, y; t)\, dy.$$

$f(x, t)$ is well defined since both $f(y)$ and $g(x, y; t)$ are in $L^2(0, \infty)$. As in the proof of Lemma 3.1, $\hat{g}_{x,t}(\lambda) = e^{-\lambda t}\phi(x, \lambda)$. Hence, by (2.10) we obtain

$$\begin{aligned} f(x, t) &= \int_0^\infty f(y)\, g(x, y; t)\, dy \\ &= \int_{-\infty}^\infty \hat{f}(\lambda)\, e^{-\lambda t}\phi(x, \lambda)\, d\rho(\lambda). \end{aligned} \tag{4.4}$$

From (4.4) and (2.4) we see that

$$\begin{aligned} \frac{\partial f(x, t)}{\partial t} &= \int_{-\infty}^\infty -\lambda \hat{f}(\lambda)\, e^{-\lambda t}\phi(x, \lambda)\, d\rho(\lambda) \\ &= \int_{-\infty}^\infty -\hat{f}(\lambda)\, e^{-\lambda t}(L_x\phi(x, \lambda))\, d\rho(\lambda) \\ &= -L_x \int_{-\infty}^\infty \hat{f}(\lambda)\, e^{-\lambda t}\phi(x, \lambda)\, d\rho(\lambda) \\ &= -L_x f(x, t). \end{aligned}$$

The boundary conditions (4.3) follow from (4.4); in particular,

$$f(x, 0) = \int_{-\infty}^\infty \hat{f}(\lambda)\, \phi(x, \lambda)\, d\rho(\lambda) = f(x).$$

By similar reasoning, we note that the function

$$g(x; t) = \int_{-\infty}^{\infty} e^{-\lambda t} \phi(x, \lambda)\, d\rho(\lambda)$$

is also a solution of Eq. (4.2). In the particular case where $\alpha = \pi/2$ in the boundary condition (2.7), $g(x; t)$ becomes the fundamental solution of (4.1) in the sense that for a given well-behaved function f.

$$\lim_{t \to 0} \int_0^{\infty} f(x)\, g(x; t)\, dx = f(0).$$

For, we have

$$\int_0^{\infty} f(x) \int_{-\infty}^{\infty} e^{-\lambda t} \phi(x, \lambda)\, d\rho(\lambda)\, dx = \int_{-\infty}^{\infty} e^{-\lambda t} \hat{f}(\lambda)\, d\rho(\lambda)$$

$$\to \int_{-\infty}^{\infty} \hat{f}(\lambda)\, d\rho(\lambda),$$

as $t \to 0$. On the other hand, since

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda)\, \phi(x, \lambda)\, d\rho(\lambda),$$

it follows that

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\lambda)\, d\rho(\lambda).$$

More details are included in [19] where we have shown that if $q(x)$ is even and $\alpha = \pi/2$, then $g(x, 0)$ is the Dirac delta function $\delta(x)$, but when $\alpha = 0$, $g(x, 0) = \delta'(x)$.

Adopting the terminology of [11, 20], we introduce two integral transforms which are closely related to Eq. (4.1). The first, the P-transform, is given for any function ψ defined on $[0, \infty)$ by

$$\tilde{\psi}(x, t) = \int_0^{\infty} \psi(y)\, g(x, y; t)\, dy, \tag{4.5}$$

whenever the integral exists; the second, the W-transform is a special case of the P-transform and is defined by

$$\begin{aligned} \tilde{\tilde{\psi}}(x) &= \tilde{\psi}(x, 1) \\ &= \int_0^{\infty} \psi(y)\, g(x, y; 1)\, dy \end{aligned} \tag{4.6}$$

whenever the integral exists.

Both transforms exist whenever $\psi(y) \in L^2(0, \infty)$ as can be seen from (i) of Lemma 3.1. The P-transform represents, within its region of convergence, the solution of the generalized heat equation (4.1) with the initial condition $\tilde{\psi}(x, 0) = \psi(x)$, while the W-transform may thus be interpreted as the value of this solution at time $t = 1$.

To establish our inversion formula, we need the following lemma.

LEMMA 4.1. *Let $\psi(y) \in L^2(0, \infty)$ and $f(x)$ be its W-transform, so that*

$$f(x) = \int_0^\infty \psi(y)\, g(x, y; 1)\, dy. \tag{4.7}$$

Then

(i) $f(x) \in L^2(0, \infty)$,

(ii) *$f(x)$ is continuous and bounded on $(0, \infty)$,*

(iii) *the generalized Fourier transform $\hat{f}(\lambda)$ of $f(x)$ is in $L^1\{(-\infty, \infty), d\rho\}$.*

Moreover,

(iv) *if $q(x)$ belongs to $C^k(0, \infty)$, then so does $f(x)$. In addition, all the derivative $f^{(j)}(x)$ $(j = 0, 1, ..., k)$ are bounded on $(0, \infty)$.*

Proof. (i) That $f(x)$ is well defined follows immediately from an application of the Cauchy–Schwartz inequality to (4.7),

$$|f(x)|^2 \leqslant \left(\int_0^\infty |\psi(y)|^2\, dy\right)\left(\int_0^\infty |g(x, y; 1)|^2\, dy\right) < \infty.$$

The last inequality follows from part (i) of Lemma 3.1. From (4.7), (3.5), and (2.10), we have that

$$f(x) = \int_{-\infty}^\infty \hat{\psi}(\lambda)\, e^{-\lambda} \phi(x, \lambda)\, d\rho(\lambda), \tag{4.8}$$

so that

$$\hat{f}(\lambda) = e^{-\lambda} \hat{\psi}(\lambda). \tag{4.9}$$

Since $\psi(y) \in L^2(0, \infty)$, then $\hat{\psi}(\lambda) \in L^2\{(-\infty, \infty), d\rho\}$ and an appeal to (2.11), (2.17), and (4.9) yields

$$\begin{aligned}\int_0^\infty |f(x)|^2\, dx &= \int_{-\infty}^\infty |\hat{f}(\lambda)|^2\, d\rho(\lambda) \\ &= \int_{-\infty}^\infty e^{-2\lambda}\, |\hat{\psi}(\lambda)|^2\, d\rho(\lambda) \\ &\leqslant e^{2b} \int_{-b}^\infty |\hat{\psi}(\lambda)|^2\, d\rho(\lambda) \\ &< \infty. \end{aligned} \tag{4.10}$$

(ii) Since $\phi(x, \lambda)$ is continuous and uniformly bounded for $x \geqslant 0$, x real, it suffices to show that the integral in (4.8) converges absolutely and uniformly for $x \geqslant 0$. This can be seen from the relation

$$\begin{aligned} |f(x)| &\leqslant c \int_{-\infty}^{\infty} e^{-\lambda} |\hat{\psi}(\lambda)| \, d\rho(\lambda) \\ &\leqslant c \left\{ \int_{-\infty}^{\infty} |\hat{\psi}(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2} \left\{ \int_{-\infty}^{\infty} e^{-2\lambda} \, d\rho(\lambda) \right\}^{1/2} \\ &< \infty, \end{aligned} \tag{4.11}$$

for some positive constant c independent of x.

(iii) The result follows from (4.9) and (4.11).

(iv) Since $q(x)$ belongs to $C^k(0, \infty)$, then so does $\phi(x, \lambda)$. Using (2.22) in (4.8) we find that, for any $0 \leqslant j \leqslant k$,

$$\begin{aligned} |f^{(j)}(x)| &\leqslant \int_{-\infty}^{\infty} |\hat{\psi}(\lambda)| \, e^{-\lambda} \, |\phi^{(j)}(x, \lambda)| \, d\rho(\lambda) \\ &\leqslant C \int_{-\infty}^{\infty} |\hat{\psi}(\lambda)| \, e^{-\lambda} \, |\lambda^p| \, d\rho(\lambda) \\ &\leqslant C \left\{ \int_{-\infty}^{\infty} |\hat{\psi}(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2} \left\{ \int_{-\infty}^{\infty} e^{-2\lambda} \, |\lambda|^p \, d\rho(\lambda) \right\}^{1/2} \\ &< \infty \end{aligned}$$

for some positive constant C independent of x. Q.E.D.

THEOREM 4.1. *Let $\psi(y) \in L^2(0, \infty)$ and $f(x)$ be its W-transform, so that*

$$f(x) = \int_0^{\infty} \psi(y) \, g(x, y; 1) \, dy.$$

Then, under the assumption of Lemma 3.2

$$\psi(y) = \lim_{t \to 1-} \int_0^{\infty} f(ax) \, g(ay, x; t) \, \eta(x, y, t) \, dx \tag{4.12}$$

in $L^2(0, \infty)$ as well as at every Lebesgue point of $\psi(y)$ uniformly in the region $0 < \delta \leqslant \operatorname{Re} t < \delta' < 1$ and $|\arg(t-1)| \geqslant \theta > \pi/2$.

Proof. From (4.8), it is clear that if x_0 is a zero of $\phi(x, \lambda)$, then it is also a zero of $f(x)$ and since both of $f(x)$ and $\phi(x, \lambda)$ are bounded (cf.

Lemma 4.1(ii)), it follows that $f(x)/\phi(x, \lambda)$ is also bounded. From the absolute convergence of (3.6), we conclude that the integral in (4.12) converges absolutely.

Let us define the operator e^{tL_x} by its power series

$$e^{tL_x} = \sum_{k=0}^{\infty} \frac{(tL_x)^k}{k!},$$

then

$$\begin{aligned} e^{tL_x}\phi(x, \lambda) &= \sum_{k=0}^{\infty} \frac{(tL_x)^k}{k!} \phi(x, \lambda) \\ &= \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} \phi(x, \lambda) \\ &= e^{(\lambda t)}\phi(x, \lambda). \end{aligned} \tag{4.13}$$

For fixed t, $0 < \operatorname{Re} t < 1$, we apply the operator e^{tL_x} to (4.8) using (4.13) and (3.6) to obtain

$$\begin{aligned} e^{tL_x}f(x) &= \int_{-\infty}^{\infty} \hat{f}(\lambda)\, e^{\lambda t}\phi(x, \lambda)\, d\rho(\lambda) \\ &= \int_{-\infty}^{\infty} \hat{f}(\lambda) \left[\int_0^{\infty} g(ax, y; t)\, \eta(x, y, t)\, \phi(ay, \lambda)\, dy \right] d\rho(\lambda) \\ &= \int_0^{\infty} g(ax, y; t)\, \eta(x, y, t) f(ay)\, dy. \end{aligned} \tag{4.14}$$

Interchanging the order of the integration is permissible since the integrals involved are absolutely convergent.

From (4.14), we obtain

$$\begin{aligned} e^{tL_x}f(x) &= \int_0^{\infty} g(ax, y; t)\, \eta(x, y, t) f(ay)\, dy \\ &= \int_0^{\infty} g(ax, y; t)\, \eta(x, y, t) \left(\int_0^{\infty} \psi(u)\, g(ay, u; 1)\, du \right) dy \\ &= \int_0^{\infty} \psi(u) \left(\int_0^{\infty} g(ax, y; t)\, g(ay, u; 1)\, \eta(x, y, t)\, dy \right) du \\ &= \int_0^{\infty} \psi(u)\, g(x, u; 1 - t)\, du \end{aligned} \tag{4.15}$$

by Lemma 3.2.

The proof is complete if we can show that

$$\lim_{\varepsilon \to 0} \int_0^{\infty} \psi(u)\, g(x, u; \varepsilon)\, du = \psi(x) \tag{4.16}$$

in $L^2(0, \infty)$ as well as at every Lebesque point of $\psi(x)$ uniformly in the region $|\arg \varepsilon| \leqslant \theta < \pi/2$.

By using Parseval's equality (2.10), (3.5), and (4.13), we obtain

$$\begin{aligned} \int_0^{\infty} \psi(u)\, g(x, u; \varepsilon)\, du &= \int_{-\infty}^{\infty} \hat{\psi}(\lambda)\, e^{-\varepsilon\lambda} \phi(x, \lambda)\, d\rho(\lambda) \\ &= \int_{-\infty}^{\infty} \hat{\psi}(\lambda)\, e^{-\varepsilon L_x} \phi(x, \lambda)\, d\rho(\lambda) \\ &= e^{-\varepsilon L_x} \psi(x). \end{aligned} \tag{4.17}$$

Finally, to show that $e^{-\varepsilon L_x} \psi(x)$ converges to $\psi(x)$ as $\varepsilon \to 0$ in the prescribed sense, we use Theorem 3 in [8] in case A, and Theorem 5 in case B, and this completes the proof. Q.E.D.

5. The Case of the Whole Real Line $(-\infty, \infty)$

Consider

$$Lu(x, \lambda) = \lambda u(x, \lambda), \qquad -\infty < x < \infty \tag{5.1}$$

$$|u(x, \lambda)| < \infty, \qquad -\infty < x < \infty. \tag{5.2}$$

Let $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be the solutions of (5.1) such that

$$\phi(x, \lambda) = 0, \qquad \phi'(x, \lambda) = -1,$$

$$\theta(x, \lambda) = 1, \qquad \theta'(x, \lambda) = 0.$$

It is known that for any $f(x) \in L^2(-\infty, \infty)$ the integrals

$$E(\lambda) = \int_{-\infty}^{\infty} \theta(x, \lambda) f(x)\, dx, \qquad F(\lambda) = \int_{-\infty}^{\infty} \phi(x, \lambda) f(x)\, dx \tag{5.3}$$

exist in the mean. Moreover, there exist three measures $d\xi(\lambda)$, $d\eta(\lambda)$, and $d\zeta(\lambda)$, independent of $f(x)$, such that the Sturm–Liouville expansion of $f(x)$ is given by

$$\begin{aligned} f(x) \sim& \int_{-\infty}^{\infty} \theta(x, \lambda)\, E(\lambda)\, d\xi(\lambda) + \int_{-\infty}^{\infty} \theta(x, \lambda)\, F(\lambda)\, d\eta(\lambda) \\ &+ \int_{-\infty}^{\infty} \phi(x, \lambda)\, E(\lambda)\, d\eta(\lambda) + \int_{-\infty}^{\infty} \phi(x, \lambda)\, F(\lambda)\, d\zeta(\lambda), \end{aligned} \tag{5.4}$$

and if $g(x) \in L^2(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

$$= \int_{-\infty}^{\infty} E(\lambda)\, E_1(\lambda)\, d\xi(\lambda) + 2 \int_{-\infty}^{\infty} (F(\lambda)\, E_1(\lambda) + F_1(\lambda)\, E(\lambda))\, d\eta(\lambda)$$

$$+ \int_{-\infty}^{\infty} F(\lambda)\, F_1(\lambda)\, d\zeta(\lambda), \tag{5.5}$$

where

$$E_1(\lambda) = \int_{-\infty}^{\infty} \theta(x, \lambda)\, g(x)\, dx, \qquad F_1(\lambda) = \int_{-\infty}^{\infty} \phi(x, \lambda)\, g(x)\, dx.$$

In the special case of (5.5) when there is a function $m(\lambda)$ such that

$$d\eta(\lambda) = m(\lambda)\, d\xi(\lambda), \qquad d\zeta(\lambda) = [m(\lambda)]^2\, d\xi(\lambda),$$

(5.2) becomes

$$f(x) \sim \int_{-\infty}^{\infty} G(\lambda)\, \psi(x, \lambda)\, d\xi(\lambda), \tag{5.6}$$

where

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\, \phi(x, \lambda),$$

and

$$G(\lambda) = \int_{-\infty}^{\infty} \psi(x, \lambda) f(x)\, dx. \tag{5.7}$$

The proof of Theorem 4.1 for $x \in (-\infty, \infty)$ is very similar to the one given for $(0, \infty)$ except that in the last statement of the proof we appeal to Propositions 2, 3 in [9] and to [8].

6. Examples

(1) Consider the operator

$$L = -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - (1/4)}{x^2}.$$

It is known that the eigenvalues are $\lambda_n = 4n + 2\alpha + 2$ and the normalized eigenfunctions are

$$\psi_n(x) = \left[\frac{2n!}{\Gamma(n+\alpha+1)}\right]^{1/2} x^{\alpha+1/2} e^{-x^2/2} L_n^\alpha(x^2),$$

where $L_n^\alpha(x)$ is the Laguerre polynomial of degree n.

It is easy to verify that the associated fuction $g(x, y; t)$ is given by

$$\begin{aligned} g_\alpha(x, y; t) &= \sum_{n=0}^{\infty} \psi_n(x)\, \psi_n(y)\, e^{-(4n+2\alpha+2)t} \\ &= 2e^{-t}(xye^{-2t})^{\alpha+1/2}\, e^{-1/2(x^2+y^2)} \\ &\quad \times \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\alpha+1)} L_n^\alpha(x^2)\, L_n^\alpha(y^2)\, e^{-4nt} \\ &= \sqrt{xy}\,(\operatorname{csch} 2t) \exp\left\{-\left(\frac{1}{2}\coth 2t\right)(x^2+y^2)\right\} I_\alpha\left(\frac{xy}{\sinh 2t}\right), \end{aligned}$$

where $I_\alpha(z)$ is the modified Bessel function of the first kind and order α. Since $q(ix) = -q(x)$, in formula (3.6) we take $a = i$ and follow the procedure given at the end of Section 3 to find that $C = i$ and $\eta(x, y, t) = (-1)^{\alpha+1} i$. Hence, noting that $J_\alpha(z)$ is the Bessel function of the first kind of order α, after some computations, we obtain

$$\begin{aligned} &\int_0^\infty g_\alpha(ix, y; t)\, \eta(x, y, t)\, \psi_n(iy)\, dy \\ &\quad = 2c_n A(t) \sqrt{x} \exp[B(t)\, x^2] \\ &\qquad \times \int_0^\infty \exp\{(\tfrac{1}{2} - B(t))\, y^2\}\, J_\alpha(2xyA(t))\, y^{\alpha+1} L_n(-y^2)\, dy \\ &\quad = c_n e^{-x^2/2} x^{\alpha+1/2} L_n(x^2)\, e^{(4n+2\alpha+2)t} \\ &\quad = \psi_n(x)\, e^{\lambda_n t}, \end{aligned} \tag{6.1}$$

where

$$c_n = \left[\frac{2n!}{\Gamma(n+\alpha+1)}\right]^{1/2},$$

$$A(t) = \tfrac{1}{2}\operatorname{csch} 2t, \qquad B(t) = \tfrac{1}{2}\coth 2t. \tag{6.2}$$

The last integral in (6.1) has been evaluated by means of formula 7.421-4 in [7].

From Theorem 4.1, we have that if

$$f(x) = 2\lambda \sqrt{x}\, e^{-\beta x^2} \int_0^\infty \sqrt{y}\, e^{-\beta y^2} I_\alpha(2\lambda xy)\, \phi(y)\, dy,$$

then

$$\phi(y) = \lim_{t \to 1^-} \frac{2}{(i)^{\alpha+1/2}} A(t) \sqrt{y}\, e^{B(t) y^2} \int_0^\infty f(ix) \sqrt{x}\, e^{-B(t) x^2} J_\alpha(2A(t)\, xy)\, dx,$$

where

$$\lambda = \lim_{t \to 1^-} A(t), \qquad \beta = \lim_{t \to 1^-} B(t).$$

Upon replacing $f(\sqrt{x})$ by $f(x)\, e^{-x/2} x^{1/2(\alpha+1/2)}$, $\phi(\sqrt{y})$ by $\phi(y) \times e^{-y/2} y^{1/2(\alpha+1/2)} e^{1/2(\alpha+1)}$, and t by $t/4$, we obtain the inversion formula for the dual Weierstrass–Laguerre transform as derived in [11].

(2) Consider the operator

$$L = -\frac{d^2}{dx^2} + x^2.$$

It is known that the eigenvalues are $\lambda_n = 2n+1$ and the normalized eigenfunctions are

$$\psi_n(x) = \left[\frac{1}{\sqrt{\pi}\, 2^n n!}\right]^{1/2} e^{-x^2/2} H_n(x),$$

where $H_n(x)$ is the Hermite polynomial of degree n.

We have that $g(x, y; t)$ is given by

$$\begin{aligned} g(x, y; t) &= \sum_{n=0}^{\infty} \psi_n(x)\, \psi_n(y)\, e^{-\lambda_n t} \\ &= \frac{1}{\sqrt{\pi}} e^{-(x^2+y^2)/2} e^{-t} \sum_{n=0}^{\infty} \frac{e^{-2nt}}{2^n n!} H_n(x)\, H_n(y) \\ &= \left(\frac{\operatorname{csch} 2t}{2\pi}\right)^{1/2} \exp\left\{\frac{xy}{\sinh 2t} - \left(\frac{1}{2} \coth 2t\right)(x^2+y^2)\right\}. \end{aligned}$$

Since $q(ix) = -q(x)$, in formula (3.6) we take $a = i$ and follow the procedure given at the end of Section 3 to find that $C = i$ and $\eta(x, y, t) = \exp\{-2ixy/\sinh 2t\}$. Therefore, if we set $d_n = \{\sqrt{n}\, 2^n n!]^{-1/2}$,

$$A(t) = \left(\frac{\operatorname{csch} 2t}{2}\right)^{1/2}, \qquad B(t) = \frac{1}{2} \coth 2t,$$

we obtain after some calculations,

$$\int_{-\infty}^{\infty} g(ix, y; t)\, \eta(x, y, t)\, \psi_n(iy)\, dy$$

$$= d_n \frac{A(t)}{\sqrt{\pi}} \exp\,[B(t)\, x^2]$$

$$\times \int_{-\infty}^{\infty} \exp\left\{ y^2\left(\frac{1}{2} - B(t)\right) - 2ixyA^2(t)\right\} H_n(iy)\, dy$$

$$= d_n e^{-x^2/2}\, H_n(x)\, e^{(2n+1)t}$$

$$= \psi_n(x)\, e^{\lambda_n t}.$$

The integral has been evaluated by means of formula 7.374–8 in [7].

The inversion formula now takes the form that if

$$f(x) = \frac{\alpha}{\sqrt{\pi}} e^{-\beta x^2} \int_{-\infty}^{\infty} \phi(y)\, e^{-\beta y^2 + 2\alpha^2 xy}\, dy, \tag{6.3}$$

then

$$\phi(y) = \frac{1}{\sqrt{\pi}} \lim_{t \to 1^-} A(t)\, e^{B(t)\, y^2} \int_{-\infty}^{\infty} f(ix) \exp\{-B(t)\, x^2 - 2i[A(t)]^2\, xy\}\, dx, \tag{6.4}$$

where

$$\alpha = \lim_{t \to 1^-} A(t), \qquad \beta = \lim_{t \to 1^-} B(t).$$

This is essentially the inversion formula for the Weierstrass transform as derived by Hirschman and Widder in [12, Theorem 4.1, Chap. 8]. For if, in Eq. (6.3), we let $y = y'/2\sqrt{\beta}$ and $x = (\sqrt{\beta}/2\alpha^2)\, x'$, we obtain

$$f\left(\frac{\sqrt{\beta}\, x'}{2\alpha^2}\right) \exp\left\{\frac{x'^2}{4}\left(\frac{\beta^2 - \alpha^4}{\alpha^4}\right)\right\} = \frac{\alpha}{2\sqrt{\pi\beta}} \int_{-\infty}^{\infty} \phi\left(\frac{y'}{2\sqrt{\beta}}\right) e^{-(x'-y')^2/4}\, dy'.$$

We note that when

$$F(x') = \exp\left\{\frac{x'^2}{4}\left(\frac{\beta^2 - \alpha^4}{\alpha^4}\right)\right\} f\left(\frac{\sqrt{\beta}\, x'}{2\alpha^2}\right) \qquad \text{and} \qquad \Phi(y') = \frac{\alpha}{\sqrt{\beta}}\, \phi\left(\frac{y'}{2\sqrt{\beta}}\right),$$

$F(x')$ is the Weierstrass transform of $\Phi(y')$ so that its inversion formula yields

$$\frac{\alpha}{\sqrt{\beta}}\, \phi\left(\frac{y'}{2\sqrt{\beta}}\right) = \lim_{u \to 1^-} \frac{1}{\sqrt{4\pi u}}\, e^{y'^2/4u}$$

$$\times \int_{-\infty}^{\infty} \exp\left\{-\frac{x'^2}{4}\left(\frac{1}{u} + \frac{\beta^2 - \alpha^4}{\alpha^4}\right) - \frac{i}{2u} x'y'\right\} f\left(\frac{i\sqrt{\beta}\, x'}{2\alpha^2}\right) dx'. \tag{6.5}$$

It follows that

$$\phi(y) = \lim_{u \to 1^-} \frac{\alpha}{\sqrt{\pi u}} e^{\beta y^2/u}$$
$$\times \int_{-\infty}^{\infty} \exp\left\{-x^2\left(\beta + \frac{\alpha^4}{\beta}\left(\frac{1}{u} - 1\right)\right) - \frac{2i\alpha^2 xy}{u}\right\} f(ix)\, dx,$$

which in the view of the definitions of α and β, is equivalent to (6.4).

(3) Consider the differential equation

$$-\frac{d^2y}{dx^2} + \frac{\alpha^2 - \frac{1}{4}}{x^2} y = s^2 y, \qquad 0 \leqslant x < \infty, \qquad \lambda = s^2.$$

It is known (cf. [15]) that, in this case, the spectrum is continuous and fills the interval $(0, \infty)$. Moreover, the role of the function $\phi(x, \lambda)$ is now played by the function $\sqrt{x}\, J_\alpha(sx)$. Formula (2.9) takes the form

$$f(x) \sim \int_0^\infty \hat{f}(s) \sqrt{x}\, J_\alpha(sx)\, s\, ds.$$

The associated function $g(x, y; t)$ is given by

$$g(x, y; t) = \int_0^\infty \sqrt{x}\, J_\alpha(sx) \sqrt{y}\, J_\alpha(sy)\, e^{-ts^2}\, s\, ds$$
$$= \sqrt{xy}\left(\frac{1}{2t}\right) \exp\left(-\frac{x^2 + y^2}{4t}\right) I_\alpha\left(\frac{xy}{2t}\right).$$

Since $q(ix) = -q(x)$, we take $a = i$ and find that $C = i$, $\eta(x, y, t) = (-1)^{\alpha + 3/2}$. Hence,

$$\int_0^\infty g(ix, y; t)\, \eta(x, y, t)\, \phi(iy, \lambda)\, dy$$
$$= \frac{\sqrt{x}}{2t} \exp\left(\frac{x^2}{4t}\right) \int_0^\infty \exp\left(-\frac{y^2}{4t}\right) I_\alpha\left(\frac{-ixy}{2t}\right) J_\alpha(isy)\, y\, dy$$
$$= \frac{\sqrt{x}}{2t} \exp\left(\frac{x^2}{4t}\right) \int_0^\infty y \exp\left(-\frac{y^2}{4t}\right) J_\alpha\left(\frac{xy}{2t}\right) I_\alpha(sy)\, dy$$
$$= \sqrt{x}\, J_\alpha(sx)\, e^{ts^2};$$

see formula 6.633–4 in [7].

By the inversion formula of Theorem 4.1, we now have that if

$$f(x)=\frac{1}{2}\sqrt{x}\,e^{-x^2/4}\int_0^\infty \psi(y)\sqrt{y}\,e^{-y^2/4}\,I_\alpha\left(\frac{xy}{2}\right)dy, \tag{6.6}$$

then

$$\psi(y)=\frac{-1}{2(i)^{\alpha+1/2}}\sqrt{y}\,e^{y^2/4}\lim_{t\to 1^-}\int_0^\infty f(ix)\sqrt{x}\,e^{-x^2/4t}J_\alpha\left(\frac{xy}{2t}\right)dx. \tag{6.7}$$

Upon replacing $\psi(y)$ by $y^{\alpha+1/2}\,\psi(y)$ and $f(x)$ by $x^{\alpha+1/2}f(x)$, we obtain the inversion formula for the Weierstrass–Hankel convolution transform as derived in [1].

References

1. F. M. Cholewinski and D. T. Haimo, The Weierstrass–Hankel convolution transform, *J. Anal. Math.* **17** (1966), 1–58.
2. F. M. Cholewinski and D. T. Haimo, The Weierstrass–Laguerre transform, *J. Math. Anal. Appl.* **23** (1968), 41–57.
3. F. M. Cholewinski and D. T. Haimo, The dual Poisson–Laguerre transform, *Trans. Amer. Math. Soc.* **144** (1969), 271–300.
4. F. M. Cholewinski and D. T. Haimo, Expansions in terms of Laguerre heat polynomials and their temperature transforms, *J. Analyse Math.* **24** (1971), 285–322.
5. F. M. Cholewinski and D. T. Haimo, Inversion of the reduced Poisson–Hankel transform, *J. Analyse Math.* **25** (1972), 323–343.
6. H. Diamond, M. Kon, and L. Raphael, Stable summation methods for a class of singular Sturm–Liouville expansion, *Proc. Amer. Math. Soc.* **81** (1981), 279–286.
7. I. Gradshteyn and I. Ryzhik, "Tables of Integrals, Series and Products," Academic Press, New York, 1965.
8. D. Gurarie, "L^p and Spectral Theory for a Class of Global Elliptic Operators," Tech. Report No. 84–10, Dept. of Math., Case Western Reserve Univ., Cleveland, Ohio, 1985.
9. D. Gurarie and M. Kon, Resolvent and regularity properties of elleptic operators, *Oper. Theory: Adv. Appl.* **11**, (1983), 151–162.
10. D. T. Haimo, Generalized temperature funtions, *Duke Math. J.* **33** (1966), 305–322.
11. D. T. Haimo, The dual Weierstrass–Laguerre transform, *Trans. Amer. Math. Soc.* **290** (1985), 597–613.
12. I. I. Hirschman and D. V. Widder, "The Convolution Transform," Princeton Univ. Press, Princeton, NJ, 1955.
13. L. Kon and L. Raphael, New multiplier methods for summing classical eigenfunction expansions, *J. Differential Equations* **50** (1983), 391–406.
14. B. Levitan and I. Sargsjan, Introduction to spectral theory, *Transl. Math. Monographs* **39** (1975).
15. E. Titchmarsh, "Eigenfunction Expansion I," Oxford Univ. Press, Oxford, 1962.
16. D. V. Widder, "An Introduction to Transform Theory," Academic press, New York, 1971.

17. A. Zayed and G. Walter, On the singularities of singular Sturm–Liouville expansions and an associated class of elliptic P.D.E.s, *SIAM J. Math. Anal.* **16** (1985), 725–740.
18. A. Zayed, Complex and real singularities of eigenfunction expansions, *Proc. Roy. Soc. Edinburgh Sect. A* **103** (1986), 179–199.
19. A. Zayed, On integral transforms whose kernels are solutions of singular Sturm–Liouville problems, *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), 201–228.
20. A. Zayed and D. T. Haimo, Inversion of an integral transform related to a general form of heat equation, *J. Math. Anal. Appl.* **131** (1988), 246–264.